JOURNAL OF APPROXIMATION THEORY **86**, 320–334 (1996)
ARTICLE NO. 0073

On Positive and Copositive Polynomial and Spline Approximation in $L_p[-1, 1]$, $0<p<\infty$

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Communicated by Dany Leviatan

Received January 9, 1995; accepted in revised form October 18, 1995

For a function $f \in \mathbf{L}_p[-1, 1]$, $0<p<\infty$, with finitely many sign changes, we construct a sequence of polynomials $P_n \in \mathbf{\Pi}_n$ which are copositive with f and such that $\|f-P_n\|_p \leqslant C\omega_\varphi(f, (n+1)^{-1})_p$, where $\omega_\varphi(f, t)_p$ denotes the Ditzian–Totik modulus of continuity in L_p metric. It was shown by S. P. Zhou that this estimate is exact in the sense that if f has at least one sign change, then ω_φ cannot be replaced by ω^2 if $1<p<\infty$. In fact, we show that even for positive approximation and all $0<p<\infty$ the same conclusion is true. Also, some results for (co)positive spline approximation, exact in the same sense, are obtained. © 1996 Academic Press, Inc.

1. INTRODUCTION

Let $\mathbf{L}_p[a, b]$ be the set of all measurable functions on $[a, b]$ such that $\|f\|_{L_p[a,b]}<\infty$, where

$$\|f\|_{L_p[a,b]} := \begin{cases} \left(\int_a^b |f(x)|^p\,dx\right)^{1/p}, & 0<p<\infty, \\ \operatorname*{ess\,sup}_{x\in[a,b]} |f(x)|, & p=\infty. \end{cases}$$

Let $\mathbf{C}^k[a, b]$ denote the set of k-times continuously differentiable functions on $[a, b]$, and $\mathbf{C}[a, b]$ the set of all continuous functions. We also denote

* Partially supported by the Izaak Walton Killam Memorial Scholarship.

0021-9045/96 $18.00

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by $\mathbf{\Pi}_n$ the set of all polynomials of degree $\leqslant n$, by $\mathcal{N}$ the set of natural numbers and $\mathcal{N}_0 := \mathcal{N} \cup \{0\}$. Throughout this paper C denote constants which are independent of f and n, and are not necessarily the same even if they occur in the same line. The notation $C = C(\mu_1, ..., \mu_v)$ is used to emphasize that C depends only on $\mu_1, ..., \mu_v$ and is independent of everything else.

For $Y_s := \{ y_1, ..., y_s \mid y_0 := -1 < y_1 < y_2 < \cdots < y_s < 1 =: y_{s+1} \}$, we denote by $\Delta^0(Y_s)$ the set of all functions $f \in \mathbf{L}_p[-1, 1]$ such that $(-1)^{s-k} f(x) \geqslant 0$ for $x \in [y_k, y_{k+1}]$, $k = 0, ..., s$, i.e., every $f \in \Delta^0(Y_s)$ has $0 \leqslant s < \infty$ sign changes at the points in Y_s and is nonnegative near 1. In particular, if $s = 0$, then $\Delta^0 := \Delta^0(Y_0)$ denotes the set of all nonnegative functions on $[-1, 1]$. A function g is said to be copositive with f if $f(x)\, g(x) \geqslant 0$, for all $x \in [-1, 1]$.

We are interested in approximating functions from $\Delta^0(Y_s)$ and Δ^0 by polynomials P_n of degree $\leqslant n$ and splines s_n with no more than n (fixed) knots that are copositive with f. If $s = 0$, this is also called *positive approximation.* For $f \in \mathbf{L}_p[-1, 1]$ let

$$E_n(f)_p := \inf_{P_n \in \mathbf{\Pi}_n} \|f - P_n\|_p$$

denote the degree of unconstrained approximation, and let

$$E_n^{(0)}(f, Y_s)_p := \inf_{P_n \in \mathbf{\Pi}_n \cap \Delta^0(Y_s)} \|f - P_n\|_p$$

be the degree of copositive approximation to f by algebraic polynomials of degree $\leqslant n$, where $\|\cdot\|_p := \|\cdot\|_{\mathbf{L}_p[-1,1]}$. In particular,

$$E_n^{(0)}(f)_p := E_n^{(0)}(f, Y_0)_p := \inf_{P_n \in \mathbf{\Pi}_n \cap \Delta^0} \|f - P_n\|_p$$

is the degree of positive approximation.

Positive approximation of $f \in \mathbf{C} \cap \Delta^0$ has the same order as that of unconstrained approximation: $E_n(f)_\infty \leqslant E_n^{(0)}(f)_\infty \leqslant 2E_n(f)_\infty$. For example, $E_n^{(0)}(f)_\infty \leqslant C\omega^m(f, n^{-1})_\infty$, $n \geqslant m-1$, where

$$\omega^m(f, t, [a, b])_p := \sup_{0 < h \leqslant t} \|\Delta_h^m(f, \cdot, [a, b])\|_{\mathbf{L}_p[a, b]}$$

denotes the usual mth modulus of smoothness of $f \in \mathbf{L}_p[a, b]$,

$$\Delta_h^m(f, x, [a, b]) := \begin{cases} \displaystyle\sum_{i=0}^{m} \binom{m}{i} (-1)^{m-i} f\left(x - \frac{m}{2} h + ih\right), \\ \qquad \text{if} \quad x \pm \dfrac{m}{2} h \in [a, b], \\ 0, \qquad \text{otherwise.} \end{cases}$$

is the symmetric mth difference and $\omega^m(f, t)_p := \omega^m(f, t, [-1, 1])_p$.

At the same time, even if f has only one sign change, following its sign is not so easy and the order of approximation deteriorates. It was shown by S. P. Zhou [21] that there exists $f \in \mathbf{C}^1[-1, 1] \cap \Delta^0(Y_s)$, $s \geqslant 1$, such that

$$\limsup_{n \to \infty} \frac{E_n^{(0)}(f, Y_s)_\infty}{\omega^4(f, n^{-1})_\infty} = +\infty. \tag{1}$$

Recently, Y. K. Hu and X. M. Yu [9] (see also [7], [8] and [16]) and K. A. Kopotun [13] showed that

$$E_n^{(0)}(f, Y_s)_\infty \leqslant C(Y_s)\, \omega^3(f, n^{-1})_\infty \tag{2}$$

and

$$E_n^{(0)}(f, Y_s)_\infty \leqslant C(Y_s)\, \omega_\varphi^3(f, n^{-1})_\infty, \tag{3}$$

respectively, where

$$\omega_\varphi^m(f, t)_p := \sup_{0 < h \leqslant t} \|\Delta_{h\varphi(\cdot)}^m(f, \cdot, [-1, 1])\|_p$$

is the mth Ditzian–Totik modulus of smoothness with $\varphi(x) := \sqrt{1 - x^2}$.

Thus, the investigation of copositive approximation of continuous functions in the uniform metric is complete (in the sense of the orders of the moduli of smoothness). At the same time, little is known about copositive approximation of functions in $\mathbf{L}_p \cap \Delta^0(Y_s)$ for $1 \leqslant p < \infty$ and $s \geqslant 1$, and it seems that nothing is known in the case for $0 < p < 1$. It turns out that things become more complicated in $\mathbf{L}_p$, and even positive approximation is no longer trivial.

It was shown by Zhou [21] that there exists $f \in \mathbf{C}^1[-1, 1] \cap \Delta^0(Y_s)$, $s \geqslant 1$ satisfying

$$\limsup_{n \to \infty} \frac{E_n^{(0)}(f, Y_s)_p}{\omega^{2 + [1/p]}(f, n^{-1})_p} = +\infty, \qquad 1 \leqslant p < \infty,$$

and that, in the case $s = 0$, there exists $f \in \mathbf{C}^1[-1, 1] \cap \Delta^0$ such that

$$\limsup_{n \to \infty} \frac{E_n^{(0)}(f)_p}{\omega^{3 + [1/p]}(f, n^{-1})_p} = +\infty, \qquad 1 \leqslant p < \infty.$$

Our first theorem shows that it is impossible to obtain the estimate in terms of ω^2 for positive polynomial approximation for all $0 < p < \infty$ (for $1 \leqslant p < \infty$ this was conjectured by Zhou [21]). The proof of this theorem, as well as those of our other main theorems, will be postponed until later sections.

THEOREM 1. *For every* $n \in \mathcal{N}$, $0 < p < \infty$, $0 < \varepsilon \leqslant 2$ *and* $A > 0$, *there exists a nonnegative function* $f \in \mathbf{C}^\infty[-1, 1]$ *such that for every polynomial* $P_n \in \mathbf{\Pi}_n$ *that is nonnegative at* $x = 1$, *the following inequality holds*:

$$\|f - P_n\|_{\mathbf{L}_p[1-\varepsilon, 1]} > A\omega^2(f, 1)_p. \tag{4}$$

Now that the order of ω^2 is impossible, we seek the next best rate. The theorem below shows that ω_φ is indeed reachable, thus, being the best order of positive polynomial approximation in $\mathbf{L}_p$.

THEOREM 2. *If* $f \in \mathbf{L}_p[-1, 1]$, $0 < p < \infty$ *and* $f(x) \geqslant 0$, $x \in [-1, 1]$, *then for every* $n \in \mathcal{N}_0$

$$E_n^{(0)}(f)_p \leqslant C\omega_\varphi(f, (n+1)^{-1})_p, \tag{5}$$

where C *is an absolute constant in the case* $1 \leqslant p < \infty$ *and* $C = C(p)$ *if* $0 < p < 1$.

Remark. It was noted by the referee that for $1 \leqslant p < \infty$ inequality (5) follows from some known results (though it seems that Theorem 2 was not explicitly stated in the literature). Namely, the polynomial operators used by K. G. Ivanov for the proof of Theorem 3 of [11] turned out to be positive. Therefore, the estimate $E_n^{(0)}(f)_p \leqslant C\tau(f, 1; \Delta_n)_{1,p}$, $1 \leqslant p < \infty$, was actually proved in [11] (we refer the reader to [11] for the definition of $\tau(f, 1; \Delta_n)_{1,p}$). Since it is rather well known that Ivanov's modulus $\tau(f, 1; \Delta_n)_{1,p}$ is equivalent to $\omega_\varphi(f, n^{-1})_p$, then (5) follows.

We also note that the above mentioned inequalities do not comprehend all the known results on positive approximation. In fact, all the estimates proved for one-sided approximation are true for positive approximation as well. We leave a more detailed discussion of this subject for some other time.

In the next theorem we show that even if f changes its sign in $(-1, 1)$, the order of approximation does not deteriorate further in comparison with positive approximation.

THEOREM 3. *Let* $Y_s = \{y_1, ..., y_s\}$ *be given, and let* $\delta := \min_{0 \leqslant i \leqslant s} |y_{i+1} - y_i|$. *If* $f \in \mathbf{L}_p[-1, 1] \cap \Delta^0(Y_s)$, $0 < p < \infty$, *then for every* $n \in \mathcal{N}_0$

$$E_n^{(0)}(f, Y_s)_p \leqslant C\omega_\varphi(f, (n+1)^{-1})_p, \tag{6}$$

where C *depends on* s, δ *and also on* p *in the case for* $0 < p < 1$.

For copositive spline approximation in the uniform norm, Hu, Leviatan and Yu [8] proved an analogue of (1) for splines with equally spaced knots and other classes of functions, and that the order of copositive approximation by splines with equal spacing is at least ω^2. Soon after, Hu and Yu [9] proved an analogue of (2) for such splines (in fact, (2) is derived from this result for splines, with the aid of results in [10]). If $p<\infty$, the rate drops to ω^1 for copositive spline approximation, too. We first state our affirmative result as a theorem below. If no continuity is desired ($r=1$), this result can be easily obtained by piecewise constant functions, see Lemma 3.5. Hu [6] proved the theorem for $r=2$ and 3, $1\leqslant p<\infty$ and equal spacing. His method can be modified for $0<p<1$ and unequal spacing. The general case (for any $r\geqslant 1$) can proved by applying Beatson's blending lemma [1] to local constant approximations of f on overlapping subintervals. To obtain such local constant approximations, one can use best constant $\mathbf{L}_p$ approximation where f does not change its sign, and use 0 where it does, (see the proof of Lemma 3.5 for error estimate). We omit the proof of the theorem.

THEOREM 4. *Let $f\in\mathbf{L}_p[-1,1]\cap\Delta^0(Y_s)$, $0<p<\infty$, $s\geqslant 0$ and let $r\geqslant 1$ be an integer. Let $\mathbf{T}_n:=\{z_0,...,z_n \mid -1:=z_0<z_1<\cdots<z_{n-1}<z_n:=1\}$ be a given knot sequence such that there are at least* $\max(2,4(r-1)^2)$ *knots in each open interval (y_j,y_{j+1}), $j=0,...,s$. Then there exists a spline $s_n\in\mathbf{C}^{r-2}[-1,1]\cap\Delta^0(Y_s)$ of order r on knot sequence $\mathbf{T}_n$ such that*

$$\|f-s_n\|_p\leqslant C\omega(f,d)_p, \tag{7}$$

where $d:=\max(z_i-z_{i-1})$ is the mesh size of $\mathbf{T}_n$, and C is a constant depending on the maximum ratio $\rho:=\max(z_{i+2}-z_{i+1})/(z_{i+1}-z_i)$ and on p in the case $0<p<1$.

Remark. The requirement that there are a certain number of knots in each interval (y_j,y_{j+1}) is not essential. It is so stated only for the sake of simple proof and notation. If it is removed, the constant C will then depend on the minimum distance δ between y_j's, which is roughly equivalent to the requirement made here.

As a direct consequence of Theorem 1, (taking $[z_{n-1},z_n]=[z_{n-1},1]$ as $[1-\varepsilon,1]$ in Theorem 1), we have our last theorem below, which says the result in (7) is the best for general nonnegative functions ($s=0$) in $\mathbf{L}_p$ approximated by nonnegative splines of any order on any knot sequences, although Hu [6] proved that ω^2 is possible if f has a nonnegative Whitney extension. (He proved this only for $1\leqslant p<\infty$ and equal spacing. But again, this can be extended to $0<p<\infty$ and arbitrary knot sequence.)

THEOREM 5. *In the case $s=0$, one can not replace $\omega(f,d)_p$ in (7) by $\omega^2(f,1)_p$, $0<p<\infty$, even if splines of any order on any given (fixed) knot sequence are used and no continuity is desired.*

2. COUNTEREXAMPLE

In this section we construct the counterexample described in Theorem 1. This counterexample is a modification of the one used by the second author in the proof of Theorem 2 of [15]. (Also, as was noted by the referee it is possible to prove Theorem 1 considering a truncated linear function $f_\delta(x) = (1-\delta-x)_+$ whose multi-fold integrals were used by A. S. Shvedov in [20].)

Proof of Theorem 1. Let $n \in \mathcal{N}$, $0 < \varepsilon \leqslant 2$, $A > 0$ and $0 < p < \infty$ be fixed, and define $f(x) := b(1-x) - \ln(1-x+e^{-b}) - \ln b$, where $b \geqslant 1$ is a parameter to be chosen later. Clearly, $f \in C^\infty[-1, 1]$, and it is easy to check that f assumes its minimum $1 - be^{-b} > 0$ at $x = 1 + e^{-b} - b^{-1} \in (0, 1)$, thus $f(x) > 0$, $x \in [-1, 1]$.

Using the estimate

$$\int_{-1}^{1} |\ln(1-x+e^{-b})|^p \, dx$$

$$= \int_{e^{-b}}^{2+e^{-b}} |\ln x|^p \, dx \leqslant \int_0^3 |\ln x|^p \, dx = \int_0^1 (-\ln x)^p \, dx + \int_1^3 (\ln x)^p \, dx$$

$$\leqslant 2(\ln 3)^p + \Gamma(p+1) =: M_1^p \tag{8}$$

we derive the inequality

$$\omega^2(f, 1)_p = \omega^2(\ln(1-x+e^{-b}), 1)_p$$
$$\leqslant 4^{\max\{1, 1/p\}} \, \|\ln(1-x+e^{-b})\|_p \leqslant 4^{\max\{1, 1/p\}} M_1 =: M_2. \tag{9}$$

Now suppose that the assertion of the theorem is not true, *i.e.*, there exists a polynomial $P_n(x) = a_0 + a_1(1-x) + \cdots + a_n(1-x)^n$ with $a_0 \geqslant 0$ such that

$$\|f - P_n\|_{\mathbf{L}_p[1-\varepsilon, 1]} \leqslant A\omega^2(f, 1)_p.$$

Then from (8) and (9), we have

$$\|P_n(x) - b(1-x) + \ln b\|_{\mathbf{L}_p[1-\varepsilon, 1]}$$
$$\leqslant 2^{\max\{1, 1/p\}} (M_1 + AM_2) =: M_3.$$

Therefore (see Lemma 7.3 of [18], for example),

$$\|(a_0 + \ln b) + (a_1 - b)(1-x) + a_2(1-x)^2 + \cdots$$
$$+ a_n(1-x)^n\|_{\mathbf{C}[1-\varepsilon, 1]} \leqslant CM_3 \varepsilon^{-1/p} =: M_4,$$

which gives

$$a_0 + \ln b \leqslant M_4 \qquad \text{or} \qquad P_n(1) = a_0 \leqslant M_4 - \ln b.$$

By choosing $b > e^{M_4}$ (note that M_4 may depend on n, ε, p and A but is independent of b) we get $P_n(1) < 0$, thus, obtaining a contradiction. ∎

3. NOTATION AND AUXILIARY RESULTS

The following notation is used in the rest of this paper:

$$\Delta_n(x) := \frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}, \qquad \delta_n(a, b) := \min\{\Delta_n(a), \Delta_n(b)\},$$

$$x_j := \cos\frac{j\pi}{n}, \qquad 0 \leqslant j \leqslant n,$$

$$I_j := [x_j, x_{j-1}], \qquad h_j := |I_j| = x_{j-1} - x_j, \qquad 1 \leqslant j \leqslant n$$

and

$$\psi_j := \psi_j(x) := \frac{h_j}{|x - x_j| + h_j}.$$

In the proof of Theorem 2 we need the following two lemmas.

LEMMA 3.1. *For every $n \in \mathcal{N}$, $1 \leqslant j \leqslant n-1$ and $\mu \geqslant 10$ there exist polynomials T_j and $\bar{T}_j$ of degree $\leqslant n$ satisfying for $x \in [-1, 1]$:*

$$\begin{aligned} 0 &\leqslant \chi_j(x) - T_j(x) \leqslant C(\mu)\, \psi_j^\mu, \\ 0 &\leqslant \bar{T}_j(x) - \chi_j(x) \leqslant C(\mu)\, \psi_j^\mu, \end{aligned} \tag{10}$$

where

$$\chi_j(x) := \begin{cases} 1, & \text{if } x \geqslant x_j, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. For the special case $\mu = 18$ polynomials T_j and $\bar{T}_j$ are, respectively, Q_j and $\bar{Q}_{j+1}$ from Lemma 1 of [14]. The general case $\mu \geqslant 10$ is similar (see also [19]). ∎

LEMMA 3.2. *For any function* $g \in \mathbf{L}_p[-1, 1]$, $0 < p < \infty$, *the following inequality holds*:

$$\left(\sum_{j=1}^{n} \omega(g, h_j, \mathscr{I}_j)_p^p \right)^{1/p} \leqslant CC_0\, \omega_\varphi(g, (n+1)^{-1})_p, \tag{11}$$

where, for every j, $\mathscr{I}_j \supset I_j$ *is such that* $|\mathscr{I}_j| \leqslant C_0\, |I_j|$, *and* C *depends on* p *if* $0 < p < 1$.

Proof. The inequality (11) for $0 < p < 1$ was proved in [2]. For $p \geqslant 1$ the proof is similar (see [12], for example). ∎

We shall make use of the next two lemmas in the proof of Theorem 3.

LEMMA 3.3. *Let* $n \in \mathscr{N}$ *be fixed, and let* S_n *be a piecewise constant spline with the knots at* x_j, $0 \leqslant j \leqslant n$. *Then for every interval* $[a, b] \subset [-1, 1]$ *the following inequality holds*:

$$\begin{aligned} &|S_n(b) - S_n(a)| \\ &\quad \leqslant C \left(1 + \frac{|b-a|}{\delta_n(a, b)} \right) \delta_n(a, b)^{-1/p}\, \omega_\varphi(S_n, n^{-1})_p, \qquad 0 < p < \infty. \end{aligned} \tag{12}$$

Proof. Let $J := \{ j \mid x_j \in [a, b] \}$. Since for every $j \in J$ the inequality $h_j = |I_j| \geqslant \delta_n(a, b)$ holds, the interval $[a, b]$ contains not more than $1 + [|b-a|/\delta_n(a, b)]$ intervals I_j . Also, for every $j \in J$ we have

$$\begin{aligned} \omega_\varphi(S_n, n^{-1})_p^p &= \sup_{0 < h \leqslant n^{-1}} \int_{-1}^{1} |\Delta_{h\varphi(x)}(S_n, x)|^p \, dx \\ &\geqslant C\, |S_n(x_j+) - S_n(x_j-)|^p\, h_j \\ &\geqslant C\, |S_n(x_j+) - S_n(x_j-)|^p\, \delta_n(a, b). \end{aligned}$$

Therefore

$$\begin{aligned} |S_n(b) - S_n(a)| &\leqslant \sum_{j \in J} |S_n(x_j+) - S_n(x_j-)| \\ &\leqslant C \left(1 + \frac{|b-a|}{\delta_n(a, b)} \right) \delta_n(a, b)^{-1/p}\, \omega_\varphi(S_n, n^{-1})_p. \end{aligned}$$ ∎

LEMMA 3.4. *For every* $y_k \in Y_s = \{ y_1, ..., y_s \}$ *and* $\mu \geqslant 2$, *there exists an increasing polynomial* $T_n(y_k, x) \in \mathbf{\Pi}_n$, *copositive with* $\operatorname{sgn}(x - y_k)$ *in* $[-1, 1]$ *and such that*

$$|\operatorname{sgn}(x - y_k) - T_n(y_k, x)| \leqslant C \left(\frac{\delta_n(y_k, x)}{|y_k - x| + \delta_n(y_k, x)} \right)^{\mu}. \tag{13}$$

Proof. The inequality was proved in [13] for $\mu = 2$ with $\Delta_n(y_k)$ instead of $\delta_n(y_k, x)$. The proof is similar for any $\mu \geqslant 2$.

Using the inequalities (see [19], for example)

$$\Delta_n(y)^2 < 4\Delta_n(x)(|x-y| + \Delta_n(x)) \tag{14}$$

and

$$\tfrac{1}{2}(|x-y| + \Delta_n(x)) < |x-y| + \Delta_n(y) < 2(|x-y| + \Delta_n(x)) \tag{15}$$

for any $x, y \in [-1, 1]$, we have

$$\frac{\Delta_n(y_k)}{|y_k - x| + \Delta_n(y_k)} \leqslant \frac{2\sqrt{\Delta_n(x)(|y_k - x| + \Delta_n(x))}}{\frac{1}{2}(|y_k - x| + \Delta_n(x))} = 4\sqrt{\frac{\Delta_n(x)}{|y_k - x| + \Delta_n(x)}}.$$

Therefore, for $T_n(y_k, x)$ satisfying

$$|\operatorname{sgn}(x - y_k) - T_n(y_k, x)| \leqslant C\left(\frac{\Delta_n(y_k)}{|y_k - x| + \Delta_n(y_k)}\right)^{2\mu},$$

we have

$$|\operatorname{sgn}(x - y_k) - T_n(y_k, x)| \leqslant C\left(\frac{\delta_n(y_k, x)}{|y_k - x| + \delta_n(y_k, x)}\right)^{\mu}. \quad \blacksquare$$

LEMMA 3.5. *Let f be as in Theorem* 4, *and* $\mathbf{T}_n$ *be such that there are at least two z_i's in each (y_j, y_{j+1}) for all j. Then there exists a piecewise constant spline s_n on $\mathbf{T}_n$ such that it is copositive with f and satisfies*

$$\|f - s_n\|_{\mathbf{L}_p(J_i)} \leqslant C\omega(f, |\mathcal{J}_i|, \mathcal{J}_i)_p \tag{16}$$

for each i, where $J_i := [z_i, z_{i+1}] \subseteq \mathcal{J}_i \subseteq [z_{i-1}, z_{i+2}]$, and C depends on the ratio $\rho := \max(z_{i+2} - z_{i+1})/(z_{i+1} - z_i)$. If $0 < p < 1$, C also depends on p. If, in particular, $z_i := x_{n-i}$, $i = 0, \ldots, n$, are used, and n is sufficiently large, then we have

$$\|f - s_n\|_p \leqslant C\omega_\varphi(f, (n+1)^{-1})_p, \tag{17}$$

where C depends on $\delta := \min_{0 \leqslant i \leqslant s} |y_{i+1} - y_i|$ and also on p in the case of $0 < p < 1$.

Proof. We call the interval $J_i := [z_i, z_{i+1}]$ contaminated if $z_i < y_j \leqslant z_{i+1}$ for some $y_j \in Y_s$, $1 \leqslant j \leqslant s$. Then by assumption there is exactly one y_j in each of the contaminated intervals J_{m_j}, $j = 1, \ldots, s$, and there is at least one non-contaminated interval J_i between J_{m_j} and $J_{m_{j+1}}$ for any $0 \leqslant j \leqslant s$.

(Here, for convenience we used the notation $m_0 := -1$, $m_{s+1} := n$, and $J_{-1} = J_n := \varnothing$.) Note that f does not change sign between J_{m_j} and J_{m_j+1}.

Let c_i be a best $\mathbf{L}_p$ constant approximation of f on J_i, $i = 0, ..., n-1$, then

$$\|f - c_i\|_{\mathbf{L}_p(J_i)} \leqslant C\omega(f, |J_i|, J_i)_p. \tag{18}$$

If J_i is not contaminated, c_i has the same sign as f. We define

$$s_n := \begin{cases} 0, & \text{for} \quad x \in [z_{m_j}, z_{m_j+1}),\ 1 \leqslant j \leqslant s, \\ c_i, & \text{for} \quad x \in [z_i, z_{i+1}),\ i \neq m_j \text{ for any } 1 \leqslant j \leqslant s. \end{cases}$$

Since c_{m_j-1} and c_{m_j+1} have opposite signs, we have

$$\begin{aligned} |c_{m_j-1}| &\leqslant |c_{m_j+1} - c_{m_j-1}| = |\mathscr{I}_{m_j}|^{-1/p} \|c_{m_j+1} - c_{m_j-1}\|_{\mathbf{L}_p(\mathscr{I}_{m_j})} \\ &\leqslant C\,|\mathscr{I}_{m_j}|^{-1/p} (\|f - c_{m_j+1}\|_{\mathbf{L}_p(\mathscr{I}_{m_j})} + \|f - c_{m_j-1}\|_{\mathbf{L}_p(\mathscr{I}_{m_j})}) \\ &\leqslant C\,|\mathscr{I}_{m_j}|^{-1/p}\, \omega(f, |\mathscr{I}_{m_j}|, \mathscr{I}_{m_j})_p, \end{aligned}$$

where $\mathscr{I}_{m_j} := [z_{m_j-1}, z_{m_j+2}]$. Here in the last step, we have used the fact that a (near) best $\mathbf{L}_p$ polynomial approximation of f on an interval I is also a near best one on an interval $J \supseteq I$ if their sizes are comparable (cf. DeVore and Popov [4, Lemma 3.3]). Hence

$$\begin{aligned} \|f - s_n\|_{\mathbf{L}_p(J_{m_j})} = \|f\|_{\mathbf{L}_p(J_{m_j})} &\leqslant C(\|f - c_{m_j-1}\|_{\mathbf{L}_p(J_{m_j})} \\ &\quad + |c_{m_j-1}|\, |J_{m_j}|^{1/p}) \\ &\leqslant C\omega(f, |\mathscr{I}_{m_j}|, \mathscr{I}_{m_j})_p, \end{aligned} \tag{19}$$

and this, together with (18), gives (16). From the construction, it is obvious that s_n is copositive with f.

For $n > N := C\delta^{-1}$ such that there are at least two x_i's in each (y_j, y_{j+1}) for all j, inequality (17) immediately follows from (16) and Lemma 3.2. ∎

4. POSITIVE POLYNOMIAL APPROXIMATION

Proof of Theorem 2. Throughout this section C denotes absolute constants in the case $1 \leqslant p < \infty$ and constants depending only on p if $0 < p < 1$. As usual, these constants are not necessarily the same even if they occur in the same line.

First, we approximate f by a piecewise constant function S:

$$S(f, x) := s_n + \sum_{j=1}^{n-1} (s_j - s_{j+1})\, \chi_j(x),$$

where s_j is a best $\mathbf{L}_p$ constant approximant to f on the interval I_j. Since, $S(f, x) = s_j$ on I_j, then $S(f, x) \geqslant 0$, $x \in [-1, 1]$. Also, it is well known (see [3], for example) that

$$\|f - s_j\|_{L_p(I_j)} \leqslant C\omega(f, h_j, I_j)_p.$$

Therefore, using Lemma 3.2 we have

$$\begin{aligned} \|f - S(f)\|_p^p &= \sum_{j=1}^{n} \int_{I_j} |f(x) - s_j|^p \, dx \\ &\leqslant C^p \sum_{j=1}^{n} \omega(f, h_j, I_j)_p^p \leqslant C^p \omega_\varphi(f, (n+1)^{-1})_p^p. \end{aligned} \tag{20}$$

Now we define

$$\begin{aligned} P_n(f, x) := s_n + \sum_{j=1}^{n-1} (s_j - s_{j+1}) \bigg(&\frac{\operatorname{sgn}(s_j - s_{j+1}) + 1}{2} \, \overline{T}_j(x) \\ &+ \frac{1 - \operatorname{sgn}(s_j - s_{j+1})}{2} \, T_j(x) \bigg). \end{aligned}$$

The polynomial $P_n(f, x)$ is nonnegative since

$$P_n(f, x) \geqslant s_n + \sum_{j=1}^{n-1} (s_j - s_{j+1}) \, \chi_j(x) = S(f, x) \geqslant 0.$$

Also, choosing $\mu := 1 + [10/\min\{1, p\}]$, employing the methods used in [2] (the case for $0 < p < 1$) and [12] ($1 \leqslant p < \infty$), and using Lemma 3.1 we obtain

$$\begin{aligned} \|P_n(f) - S(f)\|_p^p &\leqslant C^p \int_{-1}^{1} \left(\sum_{j=1}^{n-1} |s_j - s_{j+1}| \, \psi_j^\mu \right)^p dx \\ &\leqslant C^p \int_{-1}^{1} \left(\sum_{j=1}^{n-1} h_j^{-1/p} \, \|s_j - s_{j+1}\|_{\mathbf{L}_p(I_j)} \, \psi_j^\mu \right)^p dx. \end{aligned}$$

Now, using the inequality $(\sum \xi_i)^p \leqslant \sum \xi_i^p$ in the case $0 < p < 1$ and the well known Jensen inequality (for the latter the fact that $\sum_{j=1}^{n} \psi_j^\alpha \leqslant C$, $\alpha \geqslant 2$ is needed) we have

$$\begin{aligned} \|P_n(f) - S(f)\|_p^p &\leqslant C^p \sum_{j=1}^{n-1} h_j^{-1} \, \|s_j - s_{j+1}\|_{\mathbf{L}_p(I_j)}^p \int_{-1}^{1} \psi_j^{\mu \min\{1, p\}} \, dx \\ &\leqslant C^p \sum_{j=1}^{n-1} \|s_j - s_{j+1}\|_{\mathbf{L}_p(I_j)}^p, \end{aligned}$$

since $\int_{-1}^{1} \psi_j^{\alpha}\, dx \leqslant C(\alpha)\, h_j$ for $\alpha \geqslant 2$. Finally, using Minkowski's inequality for $p \geqslant 1$ and its analog if $0 < p < 1$ and the fact that s_j is a near best constant approximant to f on $I_j \cup I_{j-1}$ (here I_0 is understood as the empty set), together with Lemma 3.2, we obtain

$$\begin{aligned}\|P_n(f) - S(f)\|_p^p &\leqslant C^p \sum_{j=1}^{n} \|f - s_j\|^p_{\mathbf{L}_p(I_j \cup I_{j-1})} \\ &\leqslant C^p \sum_{j=1}^{n} \omega(f, h_j, I_j \cup I_{j-1})_p^p \leqslant C^p \omega_\varphi(f, (n+1)^{-1})_p^p.\end{aligned}$$

The proof of Theorem 2 is now complete. ▌

5. COPOSITIVE POLYNOMIAL AND SPLINE APPROXIMATION

Proof of Theorem 3. It is sufficient to prove Theorem 3 for sufficiently large n, say, $n \geqslant C\delta^{-1}$, since for small n its assertion follows from the fact that $\|f\|_p \leqslant C\omega_\varphi(f, 1)_p \leqslant C\omega_\varphi(f, (n+1)^{-1})_p$ for $f \in \Delta^0(Y_s)$, $s \geqslant 1$, and $0 \leqslant n \leqslant C\delta^{-1}$. Here, the first inequality can be proved the same way as (16), using y_j in place of z_i, and the observation that $\omega(f, 1)_p \sim \omega_\varphi(f, 1)_p$.

We shall prove Theorem 3 by induction on s, the number of sign changes. For $s = 0$ Theorem 2 gives the proof. Now we assume that (6) is valid for every function f in $\mathbf{L}_p[-1, 1] \cap \Delta^0(Y_{s-1})$ with $Y_{s-1} = \{y_1, ..., y_{s-1}\}$.

For $f \in \mathbf{L}_p[-1, 1] \cap \Delta^0(Y_s)$, it follows from Lemma 3.5 that there exists a piecewise constant spline $S \in \Delta^0(Y_s)$ with knots $\{x_j\}_{j=0}^n$ satisfying (17). Let $\bar{S}(x) := S(x) \operatorname{sgn}(x - y_s)$. Then $\bar{S} \in \mathbf{L}_p[-1, 1] \cap \Delta^0(Y_{s-1})$, and by the assumption, there exists a polynomial $\bar{P}_n \in \mathbf{\Pi}_n \cap \Delta^0(Y_{s-1})$ such that

$$\|\bar{S} - \bar{P}_n\| \leqslant C\omega_\varphi(\bar{S}, (n+1)^{-1})_p. \tag{21}$$

Define $P_n(x) := \bar{P}_n(x)\, T_n(y_s, x)$, where $T_n(y_s, x)$ is the polynomial copositive with $\operatorname{sgn}(x - y_s)$ and given in Lemma 3.4 for $\mu \geqslant 2 + 4/p$. It is apparent that $P_n \in \mathbf{\Pi}_{2n} \cap \Delta^0(Y_s)$. We are going to estimate $\|S - P_n\|_p$ and then $\|f - P_n\|_p$. From (21), we have

$$\begin{aligned}\|S - P_n\|_p &= \|\bar{S}(x) \operatorname{sgn}(x - y_s) - \bar{P}_n(x)\, T_n(y_s, x)\|_p \\ &\leqslant C\{\|\bar{S}(x)[\operatorname{sgn}(x - y_s) - T_n(y_s, x)]\|_p \\ &\quad + \|[\bar{S}(x) - \bar{P}_n(x)]\, T_n(y_s, x)\|_p\} \\ &\leqslant C(I + \omega_\varphi(\bar{S}, (n+1)^{-1})_p),\end{aligned}$$

where $I := \|\bar{S}(x)[\operatorname{sgn}(x-y_s) - T_n(y_s, x)]\|_p$. Here we have used the fact that $|T_n(y_s, x)| \leqslant C$. Since $|\bar{S}(x)| = |S(x)|$ and $S(y_s) = 0$, we have

$$I^p = \int_{-1}^{1} |S(x) - S(y_s)|^p \, |\operatorname{sgn}(x-y_s) - T_n(y_s, x)|^p \, dx.$$

It follows from Lemmas 3.3 and 3.4 that

$$\begin{aligned} I^p &\leqslant C^p \omega_\varphi(S, (n+1)^{-1})_p^p \int_{-1}^{1} \left(1 + \frac{|y_s - x|}{\delta_n(y_s, x)}\right)^p \\ &\quad \times \delta_n(y_s, x)^{-1} \left(\frac{\delta_n(y_s, x)}{|y_s - x| + \delta_n(y_s, x)}\right)^{\mu p} dx \\ &:= C^p \omega_\varphi(S, (n+1)^{-1})_p^p \, \Theta. \end{aligned}$$

And we have

$$\Theta := \int_{-1}^{1} \left(\frac{\delta_n(y_s, x)}{|y_s - x| + \delta_n(y_s, x)}\right)^{(\mu-1)p} \delta_n(y_s, x)^{-1} \, dx \leqslant C.$$

Indeed, it is easy to check the above inequality with $\Delta_n(y_s)$ instead of $\delta_n(y_s, x)$. Then, using (14) and (15), we can prove this inequality in terms of $\delta_n(y_s, x)$ as we did in the proof of Lemma 3.4.

Thus, we obtain

$$\|S - P_n\|_p \leqslant C[\omega_\varphi(S, (n+1)^{-1})_p + \omega_\varphi(\bar{S}, (n+1)^{-1})_p].$$

From (17), we have

$$\omega_\varphi(S, (n+1)^{-1})_p \leqslant C \omega_\varphi(f, (n+1)^{-1})_p,$$

and

$$\begin{aligned} \|f - P_n\|_p &\leqslant C(\|f - S\|_p + \|S - P_n\|_p) \\ &\leqslant C(\omega_\varphi(f, (n+1)^{-1})_p + \omega_\varphi(\bar{S}, (n+1)^{-1})_p). \end{aligned}$$

To complete the proof, we only need to show that

$$\omega_\varphi(\bar{S}, (n+1)^{-1})_p \leqslant C \omega_\varphi(S, (n+1)^{-1})_p.$$

Indeed, using the definition of $\bar{S}$ (more precisely, the fact that $\bar{S}$ coincides with $-S$ on $[-1, y_s]$, $|\bar{S}(x)| = |S(x)|$, and $\bar{S}$ does not change sign on $[y_{s-1}, 1]$) we conclude that for any numbers $a, b \in [-1, 1]$ such that $|b-a|$ is sufficiently small ($|b-a| \leqslant |y_s - y_{s-1}|$ will do) the following inequality holds

$$|\bar{S}(b) - \bar{S}(a)| \leqslant |S(b) - S(a)|.$$

This immediately yields

$$\begin{aligned}
&\|\Delta_{h\varphi(x)}(\bar{S}, x, [-1, 1])\|_p^p \\
&\quad = \int_{\{x \mid x \pm h\varphi(x)/2 \in [-1, 1]\}} |\bar{S}(x + h\varphi(x)/2) - \bar{S}(x - h\varphi(x)/2)|^p \, dx \\
&\quad \leqslant \int_{\{x \mid x \pm h\varphi(x)/2 \in [-1, 1]\}} |S(x + h\varphi(x)/2) - S(x - h\varphi(x)/2)|^p \, dx \\
&\quad = \|\Delta_{h\varphi(x)}(S, x, [-1, 1])\|_p^p,
\end{aligned}$$

and, therefore, completes the proof of Theorem 3. ∎

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